The Energy Equation

Kinetic $\int_V \frac{1}{2}\rho q^2 \, dv$

Potential $\int \rho \Omega \, dv$ assume $\frac{\partial \Omega}{\partial t} = 0$

Internal $\int \rho e \, dv$ $ds = 0$ no heat exchanged.

$$\frac{D}{Dt}\int_V \left\{\frac{1}{2}\rho v^2 + \rho e + \rho\Omega\right\} dV = \int_V \rho \frac{D}{Dt}\left\{\frac{1}{2}v^2 + e + \Omega\right\} dv = -\int_S p\,\vec{q}\cdot\vec{n}\, ds$$

$$= -\int_V \nabla\cdot(p\,\vec{q})\, dv$$

$$\boxed{\rho \frac{D}{Dt}\left\{\frac{1}{2}q^2 + e + \Omega\right\} = -\nabla\cdot(p\vec{q})} \quad (1)$$

On the other hand

$$\rho \frac{D\vec{q}}{Dt} = \rho\vec{F} - \nabla p$$

$$\rho\vec{q}\cdot\frac{D}{Dt}\vec{q} = \rho\vec{F}\cdot\vec{q} - \nabla p\cdot\vec{q}$$

$$\frac{\partial\Omega}{\partial t} \Rightarrow \qquad = -\rho\frac{D}{Dt}\Omega - \nabla p\cdot\vec{q}$$

$$\boxed{\rho\frac{D}{Dt}\left\{\frac{q^2}{2} + \Omega\right\} = -\vec{q}\cdot\nabla p} \quad (2)$$

(1) − (2)

$$\boxed{\rho\frac{D}{Dt}\{e\} = -p\cdot\nabla\cdot\vec{q}} \quad (3)$$

The internal energy is conserved for an Incompressible Flow

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{q}$$

$$\rho\frac{D}{Dt}e = \frac{p}{\rho}\frac{D}{Dt}\rho$$

$$\frac{D}{Dt}e = -p\frac{D}{Dt}(1/\rho)$$

$$p = \rho^{\gamma}$$

$$(\gamma-1)\frac{d\rho}{\rho} = \frac{dT}{T}$$

$$\boxed{e = -\int p\, d(1/\rho)} = R\int\frac{T\,d\rho}{\rho} = R(\gamma-1)^{-1}T = c_v T$$

$$\rho\frac{D}{Dt}(p/\rho) = \frac{Dp}{Dt} - \frac{p}{\rho}\frac{D}{Dt}\rho$$

Add (1)

$$\rho\frac{D}{Dt}\left\{\frac{p}{\rho} + e + \frac{1}{2}q^2 + \Omega\right\} = \frac{D}{Dt}p - \frac{p}{\rho}\frac{D}{Dt}\rho - \nabla\cdot p\vec{q}$$

$$= \frac{D}{Dt}p + \frac{p}{\cancel{\rho}}\nabla\cdot\vec{q} - p\nabla\vec{q} - \vec{q}\cdot\nabla p$$

$$= \frac{\partial p}{\partial t}$$

$$h = e + \frac{p}{\rho} = c_p T$$

For a steady flow :

$$\boxed{h + \frac{q^2}{2} + \Omega = \text{constant along a streamline}}$$

Second Principle in Thermodynamics

$$\frac{D}{Dt}S = 0$$

$$\left.\begin{aligned} de &= T\,ds - p\,d(1/\rho) \\ dh &= T\,ds + \frac{dp}{\rho} \end{aligned}\right\} ds = 0 \quad \left\{\begin{aligned} de &= -p\,d(1/\rho) \\ dh &= \frac{dp}{\rho} \end{aligned}\right.$$

Crocco's Equation

From Thermodynamics,

$$de = T\,ds - p\,d\left(\frac{1}{\rho}\right)$$
$$dh = T\,ds + \frac{1}{\rho}\,dp$$

Euler's Equations:

$$\vec{a} = \frac{D}{Dt}\vec{V} = \vec{f} - \frac{1}{\rho}\nabla p$$

$$\frac{D\vec{V}}{Dt} \equiv \frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}$$
$$\equiv \frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\nabla\vec{V}^2 + \vec{\xi}\times\vec{V},$$

where $\vec{\xi} = \nabla\times\vec{V}$. $\vec{\xi}$: the vorticity

$$\frac{D\vec{V}}{Dt} = \vec{f} + T\nabla s - \nabla h$$

if $\vec{f} = -\nabla\Omega$

$$\frac{\partial \vec{V}}{\partial t} + \nabla\left(h + \Omega + \frac{1}{2}\vec{V}^2\right) = T\nabla s + \vec{V}\times\vec{\xi}$$

This is Crocco's equation.

For a steady flow, $h_0 = h + \Omega + \frac{1}{2}V^2$

$$\nabla h_0 = T\nabla s + \vec{V}\times\vec{\xi}$$

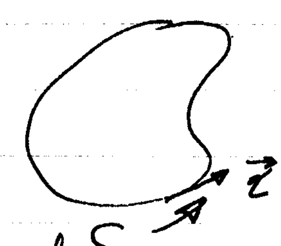
The Concept of Circulation

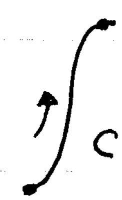
Let $\vec{V}$ be a field and C be a simply connected piecewise smooth curve, then

$$\Gamma = \int_{C^R} \vec{V} \cdot \vec{ds}$$

is the circulation of $\vec{V}$ along C.

$\vec{ds} = \vec{e}\, ds$
where $\vec{e}$ is the unit vector tangent to C, and ds is the elemental length of the arc along C. The line integral is calculated by moving along C in a given direction. If C is closed curve, the positive direction is determined by the right-hand screw rule.





What you need to calculate a circulation

1. Field $\vec{V}$
2. Curve C
3. Direction of movement along C.

What you need to calculate a circulation

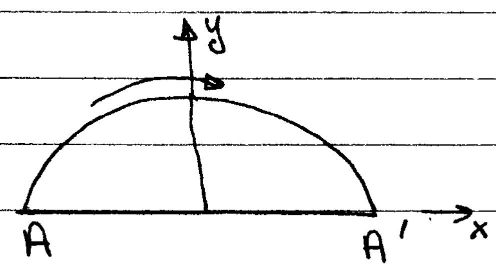
1. Field $\vec{V}$
2. Curve C
3. direction of movement along C

$\vec{V} = (x, y)$	$\vec{V} = (y, -x)$
$C: x^2 + y^2 = a^2$	$C = x^2 + y^2$
$\Gamma = 0$	$\Gamma = -a^2\theta$

Ex.

1. $\vec{V} = (y, 4x)$
2. C Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$
3. Direction $\overset{\frown}{AA'}$

$$\Gamma = \int_C \vec{V} \cdot \vec{dl}$$



$$= \int_C y\,dx + 4x\,dy$$

parametric representation

$x = a\cos\theta$ $\qquad \theta = +\pi \qquad \theta = 0$

$y = b\sin\theta$

$dx = -a\sin\theta\,d\theta$

$dy = b\cos\theta\,d\theta$

$$\Gamma = \int_{\pi}^{0} ab\left(-\sin^2\theta + 4\cos^2\theta\right) d\theta$$

$$\cos^2\theta = \frac{1}{2}\left(\cos 2\theta + 1\right)$$

$$\sin^2\theta = \frac{1}{2}\left(-\cos 2\theta + 1\right)$$

$$\Gamma = -\frac{ab}{2}\int_0^{\pi}\left[4(\cos 2\theta + 1) - (-\cos 2\theta + 1)\right] d\theta$$

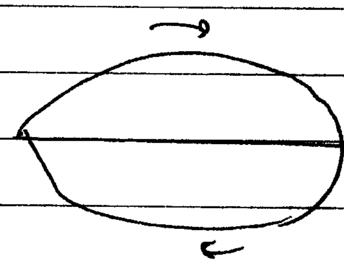
$$\Gamma = -\frac{ab}{2}\left[3\pi + 5\int_0^{\pi} \cos 2\theta \, d\theta\right]$$

$$\downarrow$$

$$\left.\frac{\sin 2\theta}{2}\right|_0^{\pi} = 0$$

$$\boxed{\Gamma = -\frac{3\pi}{2}\, ab}$$

Full ellipse



$$\Gamma = -3\pi ab$$

Change direction of motion

$$\Gamma = 3\pi ab$$

Consider $\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = 3\vec{k}$

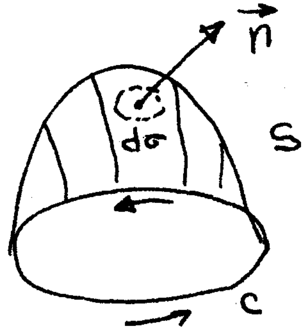
Area of ellipse $= \pi ab$

$$\Gamma = (\pi ab) \times 3 \quad \underline{\underline{?}}$$

Stokes Theorem

Consider a surface S having a closed curve C as its boundary, then,

$$\int_S (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma = \int_{C} \vec{V} \cdot d\vec{s}$$



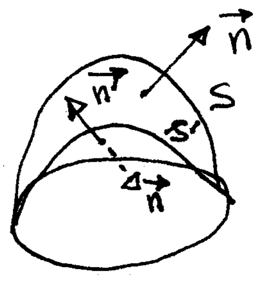
Let S' be another surface having C as a boundary,

$$\int_{S'} (\nabla \times \vec{V}) \cdot \vec{n}'\, d\sigma = \int_{C} \vec{V} \cdot d\vec{s}$$

$$\int_S (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma = \int_{S'} (\nabla \times \vec{V})\, \vec{n}'\, d\sigma$$

The volume V between S and S'

$$\int_S (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma - \int_{S'} (\nabla \times \vec{V}) \cdot \vec{n}'\, d\sigma$$



$$= \int_V \nabla \cdot (\nabla \times \vec{V})\, dv = 0$$